

Conformal symmetry for relativistic point particles: an addendum

Roberto Casalbuoni*

*Department of Physics and Astronomy,
University of Florence and INFN, Florence, Italy*

Joaquim Gomis†

*Departament d'Estructura i Constituents de la Matèria and Institut de Ciències del Cosmos,
Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain*

We extend the results of our previous work on the conformal invariant description of two relativistic point particles. We consider here the most general lagrangian by using a conformal tensor $h_{\mu\nu}$, transforming as a Wilson line, and that allows us to construct invariant expressions for velocities taken at two different space-time points.

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I. INTRODUCTION

In a previous paper [1], we have studied a conformal invariant model for two interacting relativistic particles. Our hypothesis was that the theory could be described in terms of the four-velocities of the two particles and the relative distance, assuming to form Lorentz invariant objects by contracting the vectors by using the Minkowski metric tensor. It turned out that the action was uniquely determined. In the present note we want to enlarge this framework by using a tensor $h_{\mu\nu}$, introduced by [2] (see also [3]). This tensor appears naturally in conformal field theory in the two-point function of two vector currents and transforms as a Wilson line under special conformal transformations. Therefore it is possible to construct conformal scalars by contracting the velocities of the two particles taken at different points. We will show that the more general action depends on one arbitrary function

*Electronic address: casalbuoni@fi.infn.it

†Electronic address: gomis@ecm.ub.es

of a conformal and diffeomorphism invariant dimensionless variable. The diffeomorphism (Diff) invariance is the invariance under repametrization of the single parameter describing the trajectories of the two particles.

In Section II we review some properties of the conformal group with a particular focus on the role of the discrete inversion transformation. In fact, the special conformal transformations can be recovered by using translation and inversions. In particular, we will show the transformation properties of the tensor $h_{\mu\nu}$ under inversion.

In Section III we write down the most general conformal and Diff invariant action for two particles, showing that it depends on an arbitrary function of a dimensionless variable depending on the tensor $h_{\mu\nu}$. Furthermore, we reformulate the model by introducing two einbein variables. It turns out that the interaction term depends on a function of two dimensionless variables, with one depending on the einbeins. By a general argument these two formulations are equivalent, that is eliminating the einbeins from the second formulation one is bound to recover the first one. However we have been able to show it explicitly, only in the case in which the the arbitrary function is a power in the einbeins. We show also the relation with the model presented in [1], and we introduce another very simple model using the tensor $h_{\mu\nu}$. Eventually, we evaluate the first class constraints, related to the Diff invariance, of the last two models. In particular we find that the constraint arising from the second model is quadratic in the momenta, whereas in the first one is quartic.

In Section IV we make a general study of the first class constraint associated to Diff invariance. In the general case the constraint cannot be rewritten explicitly, differently from the cases discussed in Section III. However, we are able to show the procedure that should be followed for any choice of the arbitrary function appearing in the version of the action not depending on the einbeins. We discuss also the particular case in which this function reduces to an arbitrary power. Is then possible to bring down the problem to the solution of an algebraic equation of the third degree in the dimensionless variable appearing in the arbitrary function.

Section V is devoted to the outlook and conclusions.

II. SOME PROPERTIES OF THE CONFORMAL TRANSFORMATIONS

The action of the special conformal transformations on the space-time coordinates (with a space-time of dimension different from 2) is given by

$$\bar{x}_\mu = \frac{(x^\mu + c^\mu x^2)}{1 + 2c \cdot x + c^2 x^2} = \frac{(x^\mu + c^\mu x^2)}{\sigma(x)}, \quad \sigma(x) = 1 + 2c \cdot x + c^2 x^2 \quad (1)$$

An infinitesimal transformation is given by

$$\delta x^\mu = -2(c \cdot x)x^\mu + c^\mu x^2 = x^2 h^{\mu\nu}(x) c_\nu \quad (2)$$

where

$$h^{\mu\nu}(x) = g^{\mu\nu} - 2 \frac{x^\mu x^\nu}{x^2}, \quad (3)$$

(we use the mostly plus metric $g_{\mu\nu} = (-, +, \dots, +)$), with the property

$$h^{\mu\nu}(x) h_{\nu\lambda}(x) = g_\lambda^\mu \quad (4)$$

We recall that the generic special conformal transformation can be obtained through the following combination of transformations: (inversion) \otimes (translation) \otimes (inversion), where the inversion is defined as

$$x^\mu \rightarrow \bar{x}^\mu = \frac{x^\mu}{x^2} \quad (5)$$

It follows that, in order to implement conformal invariance, it is enough to require Poincaré and inversion invariance. Let us start with the transformation properties of dx^μ . We have

$$dx^\mu \rightarrow d\bar{x}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} dx^\nu = \frac{1}{x^2} h_\nu^\mu(x) dx^\nu \quad (6)$$

From this

$$d\bar{x}^2 = \frac{dx^2}{x^4} \quad (7)$$

Then, let us consider the transformation properties of the derivative

$$\frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial \bar{x}^\mu} = \frac{\partial x^\nu}{\partial \bar{x}^\mu} \frac{\partial}{\partial x^\nu} = x^2 h_\mu^\nu(x) \frac{\partial}{\partial x^\nu} \quad (8)$$

Notice that $h_{\mu\nu}(x)$ enjoys the following property $h^{\mu\nu}(\bar{x}) = h^{\mu\nu}(x)$.

Since we are interested in a two-particle model, a quantity of interest is the relative distance $r^\mu = x_1^\mu - x_2^\mu$. Under inversion we have

$$\bar{r}^\mu = \bar{x}_1^\mu - \bar{x}_2^\mu = \frac{x_1^\mu}{x_1^2} - \frac{x_2^\mu}{x_2^2} = \frac{x_1^\mu x_2^2 - x_2^\mu x_1^2}{x_1^2 x_2^2} \quad (9)$$

From which

$$\bar{r}^2 = \frac{r^2}{x_1^2 x_2^2} \quad (10)$$

More generally, since the special conformal transformations depend on the space-time point, in order to define the scalar product among vectors defined at different points, we need the notion of parallel transport. This problem has been solved in 1970 in [2]. Let us consider the square of the distance between two points, from eq. (10) we see that under inversion r^2 has a simple scaling behavior. Then, let us consider its logarithm, which transforms as

$$\log(x_1 - x_2)^2 \rightarrow \log(x_1 - x_2)^2 - \log(x_1^2) - \log(x_2^2) \quad (11)$$

The additional terms do not contribute by taking the double derivative with respect to the coordinates of the two points. Furthermore, in order to get a dimensionless tensor we multiply the result by r^2 . That is, we define the symmetric conformal tensor

$$h_{\mu\nu}(r) = -\frac{1}{2}r^2 \frac{\partial}{\partial x_1^\mu} \frac{\partial}{\partial x_2^\nu} \log(x_1 - x_2)^2 = g_{\mu\nu} - 2\frac{r_\mu r_\nu}{r^2} \quad (12)$$

Notice that $h^{\mu\nu}(r)$ is formally equal to the h -tensor introduced previously (for this reason we use the same symbol). It follows that also $h^{\mu\nu}(r)$ satisfies the identity

$$h_{\mu\nu}(r)h^{\nu\rho}(r) = g_\mu^\rho \quad (13)$$

Let us now evaluate the properties of $h_{\mu\nu}$ under inversion. From its very definition we have

$$h_{\mu\nu}(r) \rightarrow \bar{h}_{\mu\nu}(\bar{x}) = -\frac{1}{2}\bar{r}^2 \frac{\partial}{\partial \bar{x}_1^\mu} \frac{\partial}{\partial \bar{x}_2^\nu} \log \bar{r}^2 = h_\mu^\lambda(x_1)h_\nu^\rho(x_2)h_{\lambda\rho}(r) \quad (14)$$

where we have made use of eqs. (8), (10) and (11). We see that $h_{\mu\nu}(r)$ transforms as a Wilson line, with $h_{\mu\nu}(x_1)$ and $h_{\mu\nu}(x_2)$ the holonomy factors.

Let us now define conformal contravariant vectors of weight Δ , transforming under inversion as

$$V^\mu(x) \rightarrow \bar{V}^\mu(\bar{x}) = \left(\frac{1}{x^2}\right)^\Delta h_\nu^\mu(x) V^\nu(x) \quad (15)$$

and analogous definition for a conformal covariant vector of weight Δ

$$W_\mu(x) \rightarrow \bar{W}_\mu(\bar{x}) = \left(\frac{1}{x^2}\right)^\Delta h_\mu^\nu(x) W_\nu(x) \quad (16)$$

Quite clearly dx^μ and $\partial/\partial x^\mu$ are contravariant and covariant vectors of weight 1 and -1 respectively. Due to the transformation properties of the tensor $h_{\mu\nu}(r)$, it is clear that we can use it form conformal scalar quantities, since it performs a parallel transport, that is

$$\bar{V}^\mu(\bar{x}_1)\bar{h}_{\mu\nu}(\bar{r})\bar{V}^\nu(\bar{x}_2) = \left(\frac{1}{x_1^2 x_2^2}\right)^\Delta V^\mu(x_1)h_{\mu\nu}(r)V^\nu(x_2) \equiv \left(\frac{1}{x_1^2 x_2^2}\right)^\Delta V(x_1)hV(x_2). \quad (17)$$

In particular

$$d\bar{x}_1^\mu \bar{h}_{\mu\nu}(\bar{r}) d\bar{x}_2^\nu = \frac{dx_1^\mu h_{\mu\nu}(r) dx_2^\nu}{x_1^2 x_2^2} \quad (18)$$

An analogous result for the covariant vectors of weight Δ holds.

III. GENERAL CONFORMAL INVARIANT ACTION

We start considering only the space-time variables, x_1, x_2 and their first time derivatives, and let us recall that under inversion:

$$\dot{\bar{x}}_i^2 = \frac{\dot{x}_i^2}{x_i^4}, \quad \bar{r}^2 = \frac{r^2}{x_1^2 x_2^2}, \quad \dot{\bar{x}}_1^\mu \bar{h}_{\mu\nu} \dot{\bar{x}}_2^\nu = \frac{\dot{x}_1^\mu h_{\mu\nu} \dot{x}_2^\nu}{x_1^2 x_2^2} \equiv \frac{\dot{x}_1 h \dot{x}_2}{x_1^2 x_2^2}, \quad i = 1, 2 \quad (19)$$

Since we want to require Diff invariance, we look for the most general expression transforming as a first derivative with respect to the evolution parameter τ , Furthermore we impose conformal invariance. By simple dimensional analysis we find

$$L = \left(\frac{\dot{x}_1^2 \dot{x}_2^2}{r^4}\right)^{1/4} f\left(\frac{\dot{x}_1 h \dot{x}_2}{\sqrt{\dot{x}_1^2 \dot{x}_2^2}}\right) \quad (20)$$

where f is an arbitrary function. Notice that if we choose the function f as a constant we get the lagrangian discussed in [1].

We know from [1] that a free conformal invariant action, describing a single particle depending only on space-time variables, does not exist. If one wants to describe the theory as a free term plus an interaction, then, one has to introduce the einbeins. Then, we write the free term in the form

$$L_{free} = \frac{\dot{x}_1^2}{2e_1} + \frac{\dot{x}_2^2}{2e_2} \quad (21)$$

This allows us to determine the relevant transformation properties of the einbeins. Precisely

- Under Diff the einbeins transform as time derivatives.
- Under inversion: $e_i \rightarrow e_i/x_i^4$

Then, we write down the most general invariant interaction term assuming symmetry under the exchange of the two particles. The result is

$$L_I = \left(\frac{\sqrt{e_1 e_2}}{r^2} \right) F \left(\frac{\dot{x}_1 h \dot{x}_2}{\sqrt{\dot{x}_1^2 \dot{x}_2^2}}, \frac{e_1 e_2}{\sqrt{\dot{x}_1^2 \dot{x}_2^2} r^2} \right) \quad (22)$$

where F is an arbitrary function of its two arguments which are both conformal and Diff invariant. On the other hand, if we are interested in finding an explicit relation between the two formulations, we should notice that there is a large arbitrariness in choosing the dependence of the function F from its second argument. For instance, let us consider the case in which the function F is the product of a power of the second argument times an arbitrary function of the first one, that is

$$F \left(\frac{\dot{x}_1 h \dot{x}_2}{\sqrt{\dot{x}_1^2 \dot{x}_2^2}}, \frac{e_1 e_2}{\sqrt{\dot{x}_1^2 \dot{x}_2^2} r^2} \right) = \frac{1}{2} \left(\frac{e_1 e_2}{\sqrt{\dot{x}_1^2 \dot{x}_2^2} r^2} \right)^{n-1/2} \tilde{F} \left(\frac{\dot{x}_1 h \dot{x}_2}{\sqrt{\dot{x}_1^2 \dot{x}_2^2}} \right) \quad (23)$$

The exponent $n - 1/2$ has been chosen for simplicity reasons in order to have in the expression of the corresponding L_I the einbeins to the n -th power.

The total lagrangian is

$$L = L_{free} + L_I = \frac{\dot{x}_1^2}{2e_1} + \frac{\dot{x}_2^2}{2e_2} + \frac{(e_1 e_2)^n}{2r^2} \left(\frac{1}{\sqrt{\dot{x}_1^2 \dot{x}_2^2} r^2} \right)^{n-1/2} \tilde{F} \left(\frac{\dot{x}_1 h \dot{x}_2}{\sqrt{\dot{x}_1^2 \dot{x}_2^2}} \right) \quad (24)$$

and by varying it, we get two equations for e_1 and e_2 . By solving these equations we find

$$e_1 = \left[\left(\frac{\dot{x}_1^2}{\dot{x}_2^2} \right)^n \frac{\dot{x}_1^2}{n} g \right]^{1/(2n+1)}, \quad e_2 = \left[\left(\frac{\dot{x}_2^2}{\dot{x}_1^2} \right)^n \frac{\dot{x}_2^2}{n} g \right]^{1/(2n+1)} \quad (25)$$

where we have defined

$$g = \frac{1}{r^2} \left[\frac{1}{\sqrt{\dot{x}_1^2 \dot{x}_2^2} r^2} \right]^{n-1/2} \tilde{F} \left(\frac{\dot{x}_1 h \dot{x}_2}{\sqrt{\dot{x}_1^2 \dot{x}_2^2}} \right) \quad (26)$$

Substituting the expressions for the einbeins inside the total lagrangian, we find

$$L = \frac{(1+2n)}{2} n^{-2n/(1+2n)} \left(\frac{\dot{x}_1^2 \dot{x}_2^2}{r^4} \right)^{1/4} \tilde{F} \left(\frac{\dot{x}_1 h \dot{x}_2}{\sqrt{\dot{x}_1^2 \dot{x}_2^2}} \right)^{1/(2n+1)} \quad (27)$$

and, comparing this expression with eq. (20) we get

$$\tilde{F} = \left(\frac{2}{2n+1} \right)^{2n+1} n^{2n} f^{2n+1} \quad (28)$$

This result holds also for $n = 0$, noticing that the solutions for the einbeins go to infinity, and defining $\lim_{n \rightarrow 0} n^{-2n/(1+2n)} = 1$.

We see that for any choice of n it is always possible to choose a function \tilde{F} such to reproduce the lagrangian in eq. (20). On the other hand, for an arbitrary choice of F it is practically impossible to find explicitly its relation with f . Given this arbitrariness, one can limit himself to the simplest choice $n = 1/2$. In this case the relation between f and \tilde{F} is very simple

$$\tilde{F} = \frac{1}{2}f^2 \quad (29)$$

Furtermore, it is useful to consider, two particular cases. The first one corresponds to the model described in [1], that is to the choice $f = \alpha$ and correspondingly $\tilde{F} = \frac{1}{2}\alpha^2$. Therefore $F = \frac{\alpha^2}{4}$, in agreement with [1].

We recall that in this case the Diff invariance gives rise to the constraint

$$p_1^2 p_2^2 = \frac{\alpha^4}{16} \frac{1}{r^4} \quad (30)$$

The second choice is to take f proportional to the square root of its argument, that is

$$L = \alpha \sqrt{\frac{\dot{x}_1 h \dot{x}_2}{r^2}} \quad (31)$$

By choosing F equal to his first argument (always for $n = 1/2$), that is

$$L_I = \frac{\alpha^2}{4} \left(\frac{\sqrt{e_1 e_2}}{r^2} \right) \frac{\dot{x}_1 h \dot{x}_2}{\sqrt{\dot{x}_1^2 \dot{x}_2^2}} \quad (32)$$

and eliminating the einbeins we get back the lagrangian (31). By evaluating the momenta from eq. (31) we get

$$p_1^\mu = \frac{1}{2} \frac{h^{\mu\nu} \dot{x}_{2\nu}}{\dot{x}_1 h \dot{x}_2} l, \quad p_2^\mu = \frac{1}{2} \frac{h^{\mu\nu} \dot{x}_{1\nu}}{\dot{x}_1 h \dot{x}_2} L \quad (33)$$

From these expression we obtain the constraint associated to the Diff invariance

$$p_1 h p_2 = \frac{1}{4} \frac{\alpha^2}{r^2} \quad (34)$$

IV. STUDY OF THE CONSTRAINT

In the previous Section we have discussed the constraint from the Diff invariance in two particular cases. We will discuss now the general situation. Let us start evaluating the

momenta from the lagrangian (20). They are given by the following expressions

$$\begin{aligned} p_1^\mu &= \frac{1}{2} \frac{f}{\gamma^{3/4}} \frac{\dot{x}_1^\mu \dot{x}_2^2}{r^2} + \gamma^{1/4} \frac{h^{\mu\nu} \dot{x}_{2\nu} - (\dot{x}_1 h \dot{x}_2) \dot{x}_1^\mu / \dot{x}_1^2}{\sqrt{\dot{x}_1^2 \dot{x}_2^2}} \frac{df}{d\beta} \\ p_2^\mu &= \frac{1}{2} \frac{f}{\gamma^{3/4}} \frac{\dot{x}_2^\mu \dot{x}_1^2}{r^2} + \gamma^{1/4} \frac{h^{\mu\nu} \dot{x}_{1\nu} - (\dot{x}_1 h \dot{x}_2) \dot{x}_2^\mu / \dot{x}_2^2}{\sqrt{\dot{x}_1^2 \dot{x}_2^2}} \frac{df}{d\beta} \end{aligned} \quad (35)$$

where

$$\beta = \frac{\dot{x}_1 h \dot{x}_2}{\sqrt{\dot{x}_1^2 \dot{x}_2^2}}, \quad \gamma = \frac{\dot{x}_1^2 \dot{x}_2^2}{r^4} \quad (36)$$

After some algebra we can find the following expressions

$$\begin{aligned} p_1 h p_2 &= \frac{1}{r^2} \left[\frac{1}{4} \beta f^2 + (1 - \beta^2) f \frac{df}{d\beta} - \beta(1 - \beta^2) \left(\frac{df}{d\beta} \right)^2 \right] \\ p_1^2 p_2^2 &= \frac{1}{r^4} \left[\frac{1}{4} f^2 + (1 - \beta^2) \left(\frac{df}{d\beta} \right)^2 \right]^2 \end{aligned} \quad (37)$$

Since the terms in the squared brackets depend only on β , in principle it is possible to eliminate the β dependence, for a given f , and find a relation involving $p_1 h p_2$, $p_1^2 p_2^2$ and r^2 which is the constraint that we were looking for. In practice, this cannot be done, unless we give explicitly the function $f(\beta)$. Something more can be said by taking $f = \alpha \beta^k$. Then, we get the following ratio

$$\frac{p_1 h p_2}{\sqrt{p_1^2 p_2^2}} = \pm \frac{(k - 1/2)^2 \beta^3 + k(1 - k)\beta}{(1/4 - k^2)\beta^2 - k^2} \quad (38)$$

This could be solved in β and then substituted inside one of the eqs. (37), obtaining in this way the constraint. It is quite clear the the relation between the momenta will not be polynomial. The only two cases in which we get a polynomial constraint are the ones discussed before. In the first case, $f = \alpha$ independent on β . The second equation decouples from the first one and gives directly the constraint (30). In the second particular case where we have taken $f = \alpha \sqrt{\beta}$, the β dependence disappears from the first of the eqs. (37) and we get directly the constraint (34). Also in this case the two equations decouple.

V. CONCLUSIONS AND OUTLOOK

We have constructed the most general action of two interacting relativistic particles invariant under conformal symmetry. The construction makes use of a conformal invariant tensor $h_{\mu\nu}$ that transforms as a Wilson line. This fact allow us to connect the velocities of

the two particles at different points. The lagrangian contains an arbitrary function f (see eq. (20)). We also considered the formulation of the model in terms of einbein variables.

The mass-shell constraint is general non-polynomial. However there are two particular cases of the model that have polynomial constraints, one is quartic (see eq. (30)) and was introduced in [1] and the other is quadratic (see eq. (34)).

It will be interesting to see the possible connection of these constraints with wave functions appearing in reference [4] in the study of higher spin field theories

The extension of the previous results to the case of two superparticles [5] invariant under superconformal transformations will be considered elsewhere.

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